Intermediate test Waves and Optics - 9 December 2013

Questions and answers

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This test contains 4 questions on 3 pages.

A few preliminary remarks:

- Answers may be given in Dutch.
- Please answer questions 3 & 4 on another (double) sheet of paper than questions 1 & 2.
- Put your name and student number at the top of all sheets.
- Put your student card at the edge of the desk for checking by the assistants and show it when handing in your test.

Question 1 (8 points): plane harmonic waves

A plane harmonic electromagnetic wave is specified (in SI units) by the following wave function:

$$\vec{\mathbf{E}} = \left(4\hat{\mathbf{i}} - 6\hat{\mathbf{j}}\right)\left(10^3 \text{ V/m}\right) \cos\left[\left(3x + 2y\right)\pi \times 10^7 - 1.26 \times 10^{16} t\right]$$

with $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ the unit vectors along the x- and y-axis, respectively.

Questions:

a) Draw the direction in the x-y plane along which the electric field oscillates.

b) What is the scalar value of the amplitude of the electric field ?

c) What is the direction of propagation of the wave ? Indicate this direction in the drawing used in the answer of question a).

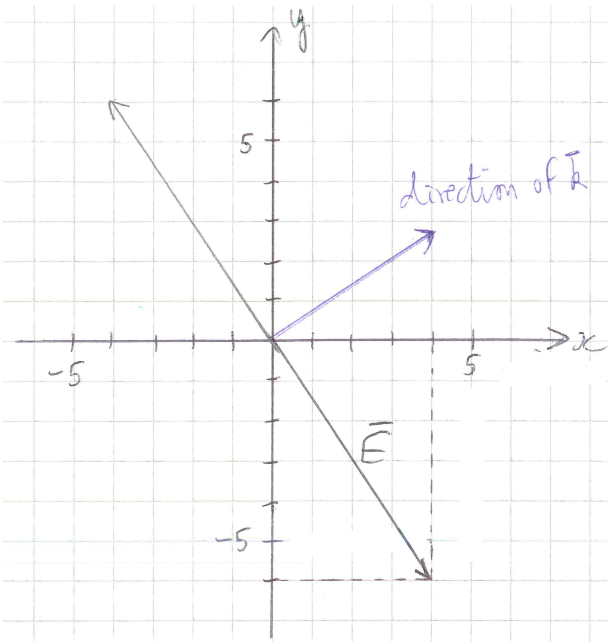
d) What is the wavelength of the wave ?

e) What is the frequency of the wave ?

Add the units to the numbers calculated.

Answer

a)



b) $$|\vec{\mathbf{E}}| = (\vec{\mathbf{E}} \bullet \vec{\mathbf{E}})^{1/2} = \left[\mathrm{E}_x^2 + \mathrm{E}_y^2 + \mathrm{E}_z^2\right]^{1/2}$$

$$= \left[4^2 + (-6)^2 + 0^2\right]^{1/2} 10^3\ \mathrm{V/m} = 7.21 \times 10^3\ \mathrm{V/m}$$

c) For a harmonic plane wave, the part of the phase containing the spatial variables is equal to $\vec{k} \bullet \vec{r}$, with $\vec{k}$ the propagation vector with direction equal to the direction of propagation of the wave. For the given wave: $\vec{k} \bullet \vec{r} = k_x x + k_y y + k_z z = (3x + 2y)\pi \times 10^7$

the direction of $\vec{k}$, and thus also the direction of propagation is given by the vector $3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$

As the wave function depends on "$\vec{k} \bullet \vec{r} - \omega t$" (minus sign !), the wave propagates in the direction indicated by the arrow in the figure.

d) wavelength $\lambda = \dfrac{2\pi}{k}$

$$k = |\vec{k}| = (\vec{k} \bullet \vec{k})^{1/2} = \pi \times 10^7 \left[3^2 + 2^2\right]^{1/2} = 3.61\pi \times 10^7\ /\mathrm{m}$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi}{3.61\pi \times 10^7} = 0.554 \times 10^{-7}\ \mathrm{m} = 55.4\ \mathrm{nm}$$

e) from the wave function: $\omega = 1.26 \times 10^{16}\ \mathrm{rad/s}$

frequency $\nu = \dfrac{\omega}{2\pi} = \dfrac{1.26 \times 10^{16}}{2\pi} = 1.91 \times 10^{15}\ /\mathrm{s}$

Question 2 (7 points): Fresnel equations, transmittance

A beam of light in air strikes the surface of a smooth piece of plastic (with index of refraction 1.5) at an angle of incidence of 30 degrees. The incident light has an electric field component parallel to the plane of incidence with an amplitude of 10.0 V/m and a component perpendicular to the plane of incidence with an amplitude of 20 V/m.

a) Calculate the **amplitude** of the **parallel** component of the electric field of the **reflected** beam.

b) Calculate the **perpendicular** component of the **transmittance**.

Make use of the appropriate Fresnel equations given below.

$$r_\perp = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad (4.34)$$

$$t_\perp = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \qquad (4.35)$$

$$r_\parallel = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad (4.40)$$

$$t_\parallel = \frac{2n_i \cos\theta_i}{n_i \cos\theta_t + n_t \cos\theta_i} \qquad (4.41)$$

For question b), in case you need to derive the appropriate equation of the transmittance, the following expression for the irradiance I of a harmonic electromagnetic wave with amplitude E_0 may be useful:

$$I = \frac{c\,\varepsilon_0}{2} E_0^2$$

Answer:

From Snell's law,

$n_i \sin\theta_i = n_t \sin\theta_t$, with $\theta_i = 30$ degrees, $n_i = 1$ and $n_t = 1.5$ follows:

$$\theta_t = \text{asin}\left[\frac{n_i}{n_t}\sin\theta_i\right] = 19.47 \text{ degrees}.$$

a) use the Fresnel equation (4.40):

$$E_{0r\parallel} = r_\parallel E_{0i\parallel} = 0.159 \times 10\,\text{V/m} = 1.59\,\text{V/m}$$

b) the perpendicular transmittance is given by:

$$T_{\perp} = \left(\frac{n_t \cos\,\theta_t}{n_i \cos\,\theta_i}\right) t_{\perp}^2 \qquad (4.63)$$

using the Fresnel equation (4.35): $T_{\perp} = 0.942$

NOTE: it is not assumed that the formule for the transmittance is known. It can be quickly derived if needed.

Question 3 (7 points): spherical and cylindrical waves

The amplitude of a spherical wave emitted by a point source is proportional to $1/r^2$, with *r* the distance to the point source.

a) Show that this relationship follows from the law of conservation of energy.

b) Use the same reasoning (thus using the law of conservation of energy) to deduce the relationship between the amplitude of a cylindrical wave and the distance *r* to the line source that generates the cylindrical wave.

NOTE: There is an error in the question. The amplitude of a spherical wave scales with $1/r$.

Answer:

The irradiance is the energy per unit surface area per unit time. So the total energy through a surface is the irradiance multiplied with the surface area (assuming constant irradiance across the surface). Conservation of energy means that the irradiance *I* multiplied with the surface area of a wavefront is constant.

a) spherical waves have spherical wavefronts, surface area = $4\pi r^2$

consider 2 wavefronts at different distances, r_1 and r_2, from the point source

conservation of energy means: $I(r_1)\ 4\pi r_1^2 = I(r_2)\ 4\pi r_2^2$ or $\dfrac{I(r_1)}{I(r_2)} = \dfrac{r_2^2}{r_1^2}$

the irradiance is proportional to the amplitude (E) squared, so $\frac{E^2(r_1)}{E^2(r_2)} = \frac{r_2^2}{r_1^2}$

or $\frac{E(r_1)}{E(r_2)} = \frac{r_2}{r_1}$, $r_1 E(r_1) = r_2 E(r_2)$ and thus $rE(r)$ is constant (independent of r), which means $E(r) \propto \frac{1}{r}$

b) The surface area of a cylindrical wavefront scales with r, the radius of the cylinder. Following the reasoning above, we obtain $\frac{I(r_1)}{I(r_2)} = \frac{r_2}{r_1}$ and

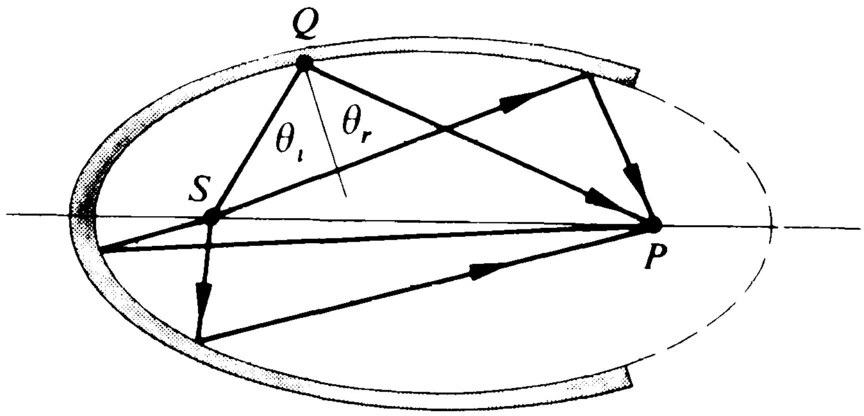
$E(r) \propto \frac{1}{\sqrt{r}}$

Question 4 (8 points): Fermat's principle

Fermat's principle allows to determine the manner in which light propagates.

a) What does the **original formulation** of Fermat's principle say about the path light will follow ? Give **two** equivalent versions. (No derivations are asked for, just state the principle.)

b) In some situations, Fermat's principle is not valid. In the lectures, an example of such a situation was discussed involving a source S and observation point P in the focii of an ellipsoid. The figure below should bring this situation to mind. For an ellipsoid, all paths SQP (with Q any point on the ellipsoid) have the same length.



Use this example to explain a situation in which Fermat's principle is not valid.

c) The existance of situations in which Fermat's principle is not valid has resulted in a **modern formulation** of Fermat's principle which is always valid. Give this modern formulation and explain its meaning.

Answer:

a)

Original formulation:
1) Light follows the fastest path, shortest time.
2) Light follows the path with the smallest optical path length.

b)

Consider a curved mirror inside the ellipsoid with the same tangent in point *Q* as the ellipsoid. This does not change anything to the path of the ray *SQP*: light going form *S* to *P* will still reflect at *Q*. Rays reflected off the curved mirror next to the point *Q* will now have a shorter optical path and thus a shorter time to go from *S* to *P*. Note that since the rays propagate in one medium only, the shortest optical path is also the shortest path. So the path *SQP* which the light follows has the longest optical path length and is thus the slowest.

c)

Light follows a path such that the optical path length is stationary with respect to variations of that path. A stationary path is a path for which the optical path length of neighbouring paths is not very different.